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Optimal Cooperative Time-Fixed Impulsive Rendezvous

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New capabilities have been added to a method that had been developed for determining optimal, i.e., minimum fuel, trajectories for the fixed-time cooperative rendezvous of two spacecraft. The method utilizes the primer vector theory. The new capabilities enable the method to accommodate cases in which there are fuel constraints on the spacecraft and/or enable the addition of a midcourse impulse to one of the vehicles' trajectories. New criteria for the addition of a midcourse impulse, analogous to the Lion and Handelsman criteria but specific to this case in which the vehicles must rendezvous at the final time, are derived. Results are presented for a large number of cases and the effect of varying parameters such as vehicle fuel constraints, vehicle initial masses, and time allowed for the rendezvous is demonstrated. Based on these results, it appears that a cooperative type of rendezvous is especially advantageous when the time allowed for rendezvous is relatively short. This may have applications in such time-critical missions as space rescue.

Nomenclature

c = magnitude of effective exhaust velocity of the rocket engine
J = cost function
m = mass
$\boldsymbol{P}$ = primer vector
R = radius of initial circular orbit
$\boldsymbol{R}$ = position vector
T = period of initial circular orbit
t = time
$\boldsymbol{v}$ = velocity vector
$\boldsymbol{x}$ = state vector
β = fuel mass fraction
Γ = reference trajectory
Γ' = perturbed trajectory
γ = initial leading angle of S/C 2 w.r.t. S/C 1
Δv = the change in velocity caused by an impulsive thrust
μ = constant of gravity
Φ_{ij} = partition of the state transition matrix, $i, j = 1, 2$, associated with a coasting trajectory

Subscripts

f = at rendezvous time
fm = corresponding to $[t_m, t_f]$
M = rendezvous type: S/C 1 has a midcourse impulse
m = rendezvous type: S/C 2 has a midcourse impulse
m = at the time when midcourse impulse is applied
$m0$ = corresponding to $[t_0, t_m]$
0 = at initial time
1 = S/C 1, except in Φ_{ij}
2 = S/C 2, except in Φ_{ij}
(1) = S/C 1
(2) = S/C 2

Introduction

THIS paper describes the continuation[1] of the work we have done on the optimal cooperative impulsive rendezvous of two spacecraft as presented in Ref. 2. In the previous work,[2] a method was developed for determining optimal trajectories for this type of rendezvous. In a cooperative impulsive rendezvous, each spacecraft makes at least one impulsive maneuver. The optimal solution consists of an optimal interception followed by an optimal rendezvous (or "terminal") maneuver. The previous work considered the simplest nontrivial case in which the intercepting trajectory of each spacecraft included only one (initial) burn and the terminal maneuver contained one burn on the part of the more efficient spacecraft at the time of interception, resulting in a total of three impulsive burns for the optimal solution. The "more efficient spacecraft" is the vehicle which, at any time, would receive the largest Δv for a given expenditure of propellant (Δm). In the present paper, more complicated cases in which the optimal trajectory of one of the spacecraft may require a midcourse impulse and/or the fuel constraint considerations may force an optimal cooperative terminal maneuver will be considered. In an optimal "cooperative" terminal maneuver both spacecraft will have to do an impulsive burn to equalize the velocity of the two vehicles. This occurs when the more efficient spacecraft at the time of interception lacks sufficient propellant to perform the terminal maneuver by itself. These new capabilities of the method raise the total number of burns allowed in an optimal solution to five.

This study continues to utilize the primer vector theory as formulated by Lawden[3] and implemented by other researchers.[4-6] An important departure from earlier analysis was discovered in the course of developing criteria for the addition of a midcourse impulse to an already optimized cooperative solution using a fixed number of impulses. Up to now, the criterion for addition of a midcourse impulse to an optimal trajectory has been known to involve a test of the primer vector magnitude associated with the trajectory.[4,5] If this magnitude exceeds unity at some point along the trajectory, addition of an impulse at that point will further lower the cost. Correspondingly, according to one of Lawden's necessary conditions for an optimal trajectory,[3] the primer vector magnitude should always be less than or equal to a constant (which may be taken to be unity) for an optimal trajectory. The equality in this condition occurs at the time of application of an impulse. Although this condition may still be true under certain circumstances in an optimal cooperative rendezvous, it has been established in this study that it is not generally valid. This is the result of the coupled

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boundary conditions at the time of rendezvous and is special to the problem of optimal cooperative rendezvous.

Analysis

Cases Without a Midcourse Impulse

The strategy adopted in this study is to start with an initial guess for the position of the interception. The initial positions and velocities of the two spacecraft and the time allowed for the rendezvous are specified. Then two Lambert problems are solved which give the trajectories of the vehicles from their specified initial positions to the rendezvous point. These "reference" trajectories are illustrated in Fig. 1 as Γ_1 and Γ_2. To complete the rendezvous, an optimal terminal maneuver is determined by applying the criteria from Prussing and Conway.[7] These criteria allocate Δv between two vehicles at interception, yielding rendezvous with the minimum expenditure of propellant. The share of the final maneuver to be performed by each vehicle is a function of the states of the vehicles at interception, particularly the masses of the vehicles, the masses of propellant aboard each vehicle, the exhaust velocities of the motors of each vehicle, and the velocities of each vehicle. Although the terminal maneuver is thus optimized, the complete rendezvous solution obtained is almost certainly nonoptimal with the cost here defined as the total mass of propellant used by both vehicles or, equivalently, the negative of the sum of the two vehicles' masses at the final time.

To optimize the cost, the primer vectors associated with each trajectory are determined. The reader is referred to Ref. 1 for the definition of the primer vector and the underlying theory behind this analysis. The gradient of the cost function may then be expressed in terms of the vehicle parameters and state variables, the two primer vectors, their time derivatives, and the state transition matrices associated with the two trajectories. This is accomplished by considering a reference rendezvous solution, as illustrated by trajectories Γ_1 and Γ_2 in Fig. 1 for the simplest case in which one vehicle (S/C 1) performs the terminal rendezvous Δv alone (noncooperatively) and in which there are no midcourse impulses on the part of either vehicle, and by perturbing this solution. The perturbed trajectories are shown as Γ_1' and Γ_2' in Fig. 1. The minus and plus signs used as superscripts modifying velocities indicate velocities immediately before and after an impulsive Δv, respectively. The cost increment is then defined as the total mass of propellant required for the three Δv's, $|v'^{+}_{10} - v'^{-}_{10}|$, $|v'^{+}_{20} - v'^{-}_{20}|$, and $|\Delta v'_{1f}|$ on the perturbed trajectory, minus the corresponding Δv's on the reference trajectory, $|v^{+}_{10} - v^{-}_{10}|$, $|v^{+}_{20} - v^{-}_{20}|$, and $\Delta v_{1f}|$. The calculation of the cost increment will then lead to an expression for the cost gradient. The independent variables for this case are the times of initial burns by the two vehicles, t_{10} and t_{20}; and the time and position of the terminal maneuver, t_f and $\boldsymbol{r}_f$, respectively.

As mentioned earlier, depending on whether the fuel constraint is active at the time of the terminal maneuver, this maneuver may be performed cooperatively or just by one vehicle. This leads to four different cases, rendezvous types 1–4 in Table 1, each with its own expression for the cost gradient. (Two other possibilities for the rendezvous, types 5 and 6, are completely noncooperative, i.e., one vehicle in each case is passive so that the final orbit for the two vehicles is the initial orbit of the passive vehicle.) In two of these cases (types 1 and 2) there is no fuel constraint at interception and one of the vehicles, namely, the more efficient one, will perform the entire terminal maneuver. In the other two cases (types 3 and 4), because of the fuel constraint, the more efficient vehicle will contribute to the terminal maneuver by exhausting all of its remaining propellant and the rest of the terminal maneuver will be performed by the other spacecraft.

It is observed that the expressions for the cost gradient in any of the given two pairs of cases are essentially the same with roles of the two spacecraft switching. As an example, the cost gradients for two cases are given here. (Gradients for all cases are given in Ref. 1.)

Equation (1) is the cost gradient for the case in which there is no fuel constraint at the time of the terminal maneuver and spacecraft 1 is the more efficient vehicle (case 1). Therefore, spacecraft 1 performs the terminal maneuver.

$$[\nabla(-m_f)]^T = \begin{bmatrix} \alpha_1 \dot{\mathbf{P}}_{10}^{+T} \Delta\mathbf{v}_{10} \\ [\alpha_1 \mathbf{P}_{1f}^T (\Phi_{21} - \Phi_{22}\Phi_{12}^{-1}\Phi_{11})^{(2)}_{(t_f,\,t_0)} + \alpha_2[\dot{\mathbf{P}}_{20}^{+T} + \mathbf{P}_{2f}^T(\Phi_{21} - \Phi_{22}\Phi_{12}^{-1}\Phi_{11})^{(2)}_{(t_f,\,t_0)}]]\Delta\mathbf{v}_{20} \\ \alpha_1[\mathbf{P}_{1f}^T(\Phi_{22}\Phi_{12}^{-1})^{(2)}_{(t_f,\,t_0)}\mathbf{v}_{2f}^- - \dot{\mathbf{P}}_{1f}^{-T}\mathbf{v}_{1f}^-] + \alpha_2[\mathbf{P}_{2f}^T(\Phi_{22}\Phi_{12}^{-1})^{(2)}_{(t_f,\,t_0)}\mathbf{v}_{2f}^- - \dot{\mathbf{P}}_{2f}^{-T}\mathbf{v}_{2f}^-] \\ [\alpha_1[\dot{\mathbf{P}}_{1f}^{-T} - \mathbf{P}_{1f}^T(\Phi_{22}\Phi_{12}^{-1})^{(2)}_{(t_f,\,t_0)}] + \alpha_2[\dot{\mathbf{P}}_{2f}^{-T} - \mathbf{P}_{2f}^T(\Phi_{22}\Phi_{12}^{-1})^{(2)}_{(t_f,\,t_0)}]]^T \end{bmatrix} \tag{1}$$

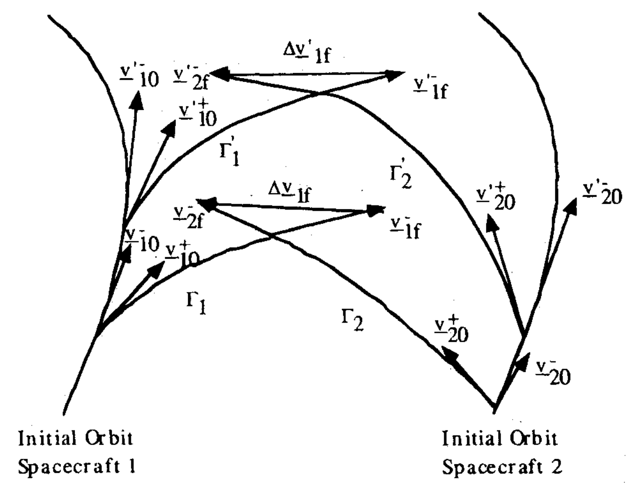

Fig. 1 Reference and perturbed trajectories of the two vehicles for rendezvous case 1 in which spacecraft 1 performs the terminal maneuver.

where

$$\alpha_1 = \frac{m_{10}}{c_1}\exp\left[-\frac{(|\Delta\mathbf{v}_{10}| + |\Delta\mathbf{v}_{1f}|)}{c_1}\right] \tag{2}$$

$$\alpha_2 = \frac{m_{20}}{c_2}\exp\left[-\frac{|\Delta\mathbf{v}_{20}|}{c_2}\right] \tag{3}$$

$$\mathbf{x} = \begin{bmatrix} t_{10} \\ t_{20} \\ t_f \\ \mathbf{r}_f \end{bmatrix} \tag{4}$$

The partitions of the state transition matrix Φ are defined by the variational equation for the primer vector as

$$\begin{bmatrix} \mathbf{P}(t) \\ \dot{\mathbf{P}}(t) \end{bmatrix} = \begin{bmatrix} \Phi_{11}(t, t_0) & \Phi_{12}(t, t_0) \\ \Phi_{21}(t, t_0) & \Phi_{22}(t, t_0) \end{bmatrix} \begin{bmatrix} \mathbf{P}(t_0) \\ \dot{\mathbf{P}}(t_0) \end{bmatrix} \tag{5}$$

The state transition matrix may be found in Ref. 8 and in the

Table 1 Definition of the different types of rendezvous

Rendezvous type	Description
1	S/C 1 performs the terminal maneuver
2	S/C 2 performs the terminal maneuver
3	S/C 1 exhausts its fuel in the course of performing the terminal maneuver and S/C 2 takes part in the terminal maneuver
4	S/C 2 exhausts its fuel in the course of performing the terminal maneuver and S/C 1 takes part in the terminal maneuver
5	S/C 1 is active and S/C 2 is passive
6	S/C 2 is active and S/C 1 is passive
M	Rendezvous solution includes a midcourse impulse on the trajectory of S/C 1
m	Rendezvous solution includes a midcourse impulse on the trajectory of S/C 2

Appendix to this paper. The superscript (2) in Eq. (1) indicates that the partitions of the transition matrix are those appropriate to the trajectory of spacecraft (2).

Equation (6) is the cost gradient for the case in which spacecraft 1 is the more efficient vehicle. However, because of the fuel constraint imposed on it, the optimal terminal maneuver also involves an impulsive burn by spacecraft 2.

$$B_2 = B_1 - I_3 \tag{12}$$

where $I_3 = 3 \times 3$ identity matrix, and

$$B_4^{(k)} = (\Phi_{22}\,\Phi_{12}^{-1})^{(k)}_{(t_f,\ t_0)} \tag{13}$$

The expressions for the cost gradient are provided to an optimization routine which uses a gradient search method to calculate the optimal solution. The process starts by choosing an arbitrary position for the rendezvous at the specified final time t_f. The values of t_{10} and t_{20} are set at zero. The trajectory that results from solving the Lambert problem for each vehicle will serve as the reference trajectory similar to the one illustrated in Fig. 1. Optimizing the terminal maneuver for this reference trajectory will indicate the proper choice of cost gradient to be used by the optimization routine. The routine determines the best direction to move the variables to lower the cost by calculating the cost gradient for the reference trajectory. The trajectories based on the new values of the variables will become the new reference trajectories and so the search continues.

Cases with a Midcourse Impulse

After the optimal solution is reached, the criterion for the addition of a midcourse impulse, to be described shortly, will be applied to the two spacecrafts' optimized trajectories. If addition of a midcourse impulse further lowers the cost, new variables, the time t_m and position $\boldsymbol{R}_m$ of the new impulse, are added to the list of the independent variables of the problem. A new set of expressions for the cost gradient is then needed for this new configuration. Figure 2 shows the reference rendezvous solution and the perturbed solution after adding a midcourse impulse to trajectory Γ_1. The midcourse impulse could be added to the trajectory of either vehicle (the case in which both vehicles' trajectories include a midcourse impulse is not considered in this study), and for each of these cases there are four different possibilities for the optimal terminal maneuver, resulting in a total of eight different cost gradients (corresponding to cases $1m$, $1M$, $2m, \ldots, 4M$ of Table 1). Once again, there are essentially only four different expressions for the cost gradient with the role of the two vehicles reversed to yield the other four expressions. As an example, the cost gradient for the case in which spacecraft 1 performs the terminal maneuver alone but which includes a midcourse impulse on the trajectory of spacecraft 2 is given in Eq. (14).

$$\text{ob}\nabla(-m_f)]^T = \begin{bmatrix} \{\alpha_1\dot{\mathbf{P}}_{10}^{+T} + (\alpha_1-\alpha_2)\,\mathbf{P}_{1f}^T[\mathbf{A}_1\dot{\mathbf{P}}_{10}^{+T} - (-\mathbf{A}_1\,\mathbf{P}_{1f}^T + \mathrm{B}_2)(\Phi_{21} - \Phi_{22}\,\Phi_{12}^{-1}\,\Phi_{11})^{(1)}_{(t_f,\ t_0)}]\}\Delta\mathbf{v}_{10} \\ [\alpha_2\dot{\mathbf{P}}_{20}^{+T} + (\alpha_1-\alpha_2)\,\mathbf{P}_{1f}^T\,B_1\,(\Phi_{21} - \Phi_{22}\,\Phi_{12}^{-1}\,\Phi_{11})^{(2)}_{(t_f,\ t_0)}]\,\Delta\mathbf{v}_{20} \\ -\alpha_1\dot{\mathbf{P}}_{1f}^{-T}\,\mathbf{v}_{1f}^- - \alpha_2\dot{\mathbf{P}}_{2f}^{-T}\,\mathbf{v}_{2f}^- + (\alpha_1-\alpha_2)\,\mathbf{P}_{1f}^T\,\{[\mathbf{A}_1\mathbf{A}_2^T - B_2\,B_4^{(1)}]\mathbf{v}_{lf}^- + B_1B_4^{(2)}\mathbf{v}_{2f}^-\} \\ \alpha_1\,\dot{\mathbf{P}}_{1f}^{-T} + \alpha_2\,\dot{\mathbf{P}}_{2f}^{-T} - (\alpha_1-\alpha_2)\,\mathbf{P}_{1f}^T[\mathbf{A}_1\mathbf{A}_2^T - B_2\,B_4^{(1)}] + B_2B_4^{(2)}] \end{bmatrix} \tag{6}$$

where

$$\alpha_1 = \frac{m_{10}}{c_1}\exp\left[-\frac{(|\Delta\mathbf{v}_{10}|+|\Delta\mathbf{v}_{1f}|)}{c_1}\right] \tag{7}$$

$$\alpha_2 = \frac{m_{20}}{c_2}\exp\left[-\frac{(|\Delta\mathbf{v}_{20}|+|\Delta\mathbf{v}_{2f}|)}{c_2}\right] \tag{8}$$

$$\mathbf{A}_1 = \frac{(m_{10})^2\,(1-\beta_{10}^-)\exp[|-\Delta\mathbf{v}_{10}|/c_1]}{(m_{1f}^-)^2\,(1-\beta_{1f}^-)}\,\mathbf{P}_{1f} \tag{9}$$

$$\mathbf{A}_2^T = \mathbf{P}_{1f}^T\,(\Phi_{22}\,\Phi_{12}^{-1})^{(1)}_{(t_f,\ t_0)} - \dot{\mathbf{P}}_{1f}^{-T} \tag{10}$$

$$B_1 = \frac{c_1\ln(1-\beta_{1f}^-)[\mathbf{P}_{1f}\mathbf{P}_{1f}^T - I_3]}{k} \tag{11}$$

where β_{1f}^- is spacecraft 1's fuel mass fraction at interception,

$$[\nabla(-m_f)]^T = \begin{bmatrix} \alpha_1\dot{\mathbf{P}}_{10}^{+T}\,\Delta\mathbf{v}_{10} \\ [\alpha_1\mathbf{P}_{1f}^T(\Phi_{21} - \Phi_{22}\,\Phi_{12}^{-1}\,\Phi_{11})^{(2)}_{(t_f,\ t_0)} + \alpha_2[\dot{\mathbf{P}}_{20}^{+T} + \mathbf{P}_{2f}^T(\Phi_{21} - \Phi_{22}\,\Phi_{12}^{-1}\,\Phi_{11})^{(2)}_{(t_f,\ t_0)}]]\Delta\mathbf{v}_{20} \\ \alpha_1[\mathbf{P}_{1f}^T(\Phi_{22}\,\Phi_{12}^{-1})^{(2)}_{(t_f,\ t_0)}\,\mathbf{v}_{2f}^- - \dot{\mathbf{P}}_{1f}^{-T}\,\mathbf{v}_{1f}^-] + \alpha_2[\mathbf{P}_{2f}^T\,(\Phi_{22}\,\Phi_{12}^{-1})^{(2)}_{(t_f,\ t_0)}\mathbf{v}_{2f}^- - \dot{\mathbf{P}}_{2f}^{-T}\,\mathbf{v}_{2f}^-] \\ [\alpha_1[\dot{\mathbf{P}}_{1f}^{-T} - \mathbf{P}_{1f}^T\,(\Phi_{22}\,\Phi_{12}^{-1})^{(2)}_{(t_f,\ t_0)}] + \alpha_2[\dot{\mathbf{P}}_{2f}^{-T} - \mathbf{P}_{2f}^T\,(\Phi_{22}\,\Phi_{12}^{-1})^{(2)}_{(t_f\ t_0)}]]^T \\ \alpha_1(\dot{\mathbf{P}}_{1m}^{+T}\,\mathbf{v}_{1m}^+ - \dot{\mathbf{P}}_{1m}^{-T}\,\mathbf{v}_{1m}^-) \\ \alpha_1(\dot{\mathbf{P}}_{1m}^{-T} - \dot{\mathbf{P}}_{1m}^{+T}) \end{bmatrix} \tag{14}$$

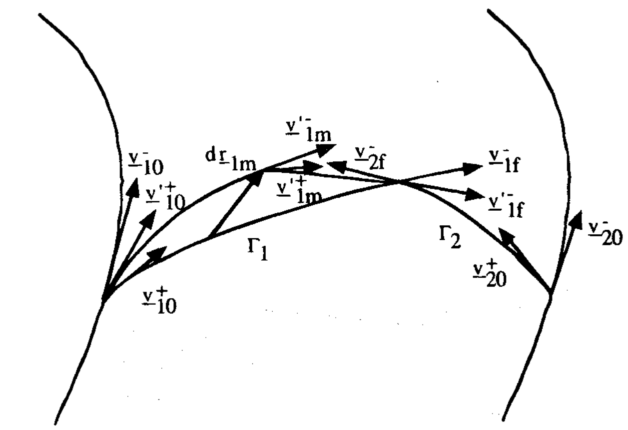

Fig. 2 Reference and perturbed trajectories of the two vehicles for rendezvous case 1M showing the addition of a midcourse impulse to spacecraft 1's trajectory.

where

$$\alpha_1 = \frac{m_{10}}{c_1}\exp\left[-\frac{(|\Delta\mathbf{v}_{10}|+|\Delta\mathbf{v}_{1m}|+|\Delta\mathbf{v}_{1f}|)}{c_1}\right] \tag{15}$$

$$\alpha_2 = \frac{m_{20}}{c_2}\exp\left[-\frac{|\Delta\mathbf{v}_{20}|}{c_2}\right] \tag{16}$$

$$\mathbf{x} = \begin{bmatrix} t_{10} \\ t_{20} \\ t_f \\ \mathbf{r}_f \\ t_{1m} \\ \mathbf{r}_{1m} \end{bmatrix} \tag{17}$$

The procedure for obtaining the terminal solution after adding a midcourse impulse is the same as before. Starting with a reference rendezvous solution, which for the most part is the same as the already optimized solution without a midcourse impulse, the optimization routine is provided with the appropriate cost gradient expression to update the values of the problem's independent variables: t_{10}, t_{20}, t_f, $\mathbf{r}_f$, t_m, and $\mathbf{r}_m$, until the optimal solution is reached.

Another possibility that may arise in the course of search for an optimal solution is that the optimal solution may tend to converge to an active/passive type of rendezvous. In contrast to the cooperative rendezvous, in an active/passive rendezvous one of the spacecraft performs all of the maneuvers (cases 5 and 6 of Table 1). Should this situation occur, the optimization routine is provided with a cost gradient for the active/passive rendezvous. On reaching the optimal solution (a two-impulse trajectory), the criterion for adding a midcourse impulse is applied to this case as well. A separate cost gradient is then used if a three-impulse solution further lowers the cost. Figure 3 summarizes the algorithm just described to solve for the optimal cooperative rendezvous.

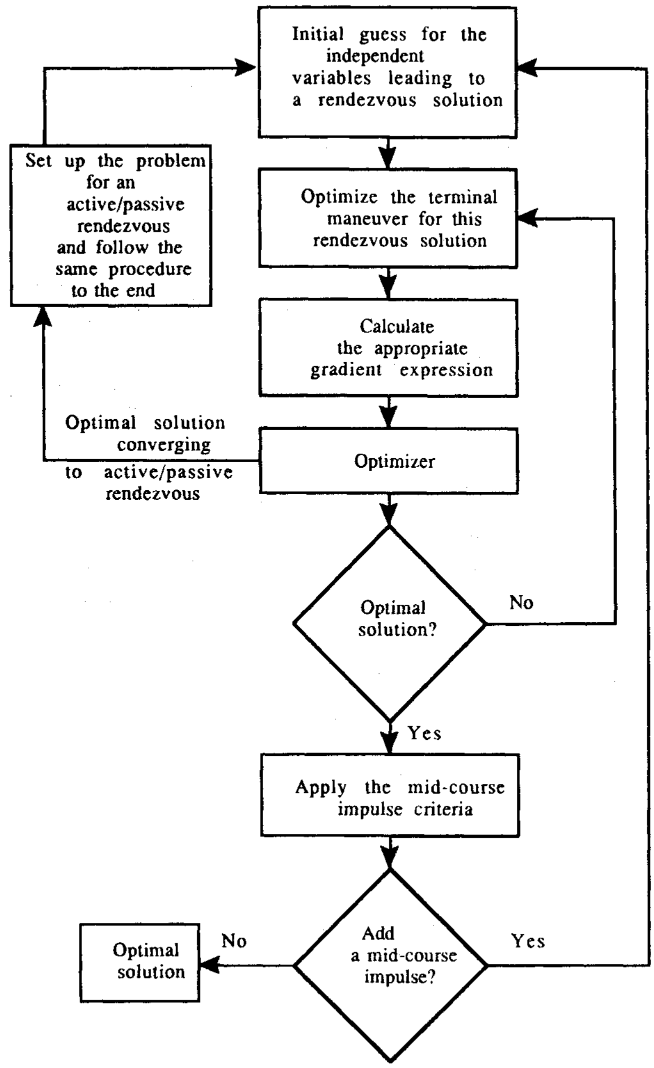

Fig. 3 Flowchart of the optimization algorithm.

Criteria for Addition of a Midcourse Impulse

The previous section describes the derivation and use of the gradient of the cost for the case in which it is optimal to have a midcourse impulse in the trajectory of one of the two spacecraft. How is it determined that a midcourse impulse would reduce the final cost?

One of Lawden's necessary conditions for an optimal trajectory[3] states that the primer vector magnitude associated with an optimal trajectory is less than or equal to a constant value which may be taken to be unity. The equality occurs at the time of an impulse. Lion and Handelsman[4] established the criterion regarding when the addition of a midcourse impulse to a trajectory will further lower the cost. The analysis leading to this switching function is similar to that described earlier for the derivation of the cost gradient. A two-impulse reference trajectory is considered and the increment in cost, $\mathrm{d}J$, is calculated after adding an arbitrary midcourse impulse with an infinitesimally small $\Delta \boldsymbol{v}$ to this reference trajectory (as in Fig. 2). It may be shown[4] then that

$$\mathrm{d}J \propto (1 - |\mathbf{P}_m|) \tag{18}$$

where $\boldsymbol{P}_m$ is the primer vector along the reference two-impulse trajectory at the time the midcourse impulse is added.

It is obvious from relation (18) that addition of a midcourse impulse will lower the cost (i.e., $\mathrm{d}J < 0$) only if $|\boldsymbol{P}_m| > 1$. Therefore, if the primer vector magnitude is larger than unity along a trajectory, that trajectory cannot be an optimal trajectory in the sense that addition of a midcourse impulse at a point where $|\boldsymbol{P}_m| > 1$ will further lower the cost.

The same approach was used in this work to derive the criteria for addition of a midcourse impulse when the trajectories include a cooperative impulsive rendezvous. There is a total of eight different cases depending on the nature of the optimal terminal maneuver and to which spacecraft's trajectory the midcourse impulse is applied. However, by switching the roles of the two spacecraft in the analysis, only four cases need to be considered. These four cases are as follows.

1) Spacecraft 1 performs the terminal maneuver. The midcourse impulse is added to spacecraft 1's trajectory, case $1M$.

2) Spacecraft 1 performs the terminal maneuver. The midcourse impulse is added to spacecraft 2's trajectory, case $1m$.

3) Spacecraft 1 exhausts its fuel in the course of performing the terminal maneuver and spacecraft 2 contributes to the terminal maneuver. The midcourse impulse is added to spacecraft 1's trajectory, case $3M$.

4) Spacecraft 1 exhausts its fuel in the course of performing the terminal maneuver and spacecraft 2 contributes to the terminal maneuver. The midcourse impulse is added to spacecraft 2's trajectory, case $3m$.

Of all of the cases considered, only case $1M$ yielded the same switching function for the primer vector magnitude as represented by relation (18). For instance, the corresponding relation for case $1m$ is as follows:

$$\mathrm{d}J \propto [(A_1 - A_2)\, z^T - A_2\, P_{1m}^T]\, \eta + A_1 \tag{19}$$

where η is the unit vector in the direction of the midcourse impulse,

$$z^T = P_{10}^T\, \Phi_{12(1m0)}^{-1}\, [(\Phi_{22}\, \Phi_{12}^{-1})_{(1mf)} - (\Phi_{22}\, \Phi_{12}^{-1})_{(1m0)}]^{-1} \tag{20}$$

$$A_1 = \frac{m_{10}}{c_1} \exp\left[\frac{|\Delta \mathbf{v}_{10}| + |\Delta \mathbf{v}_{20}|}{c_1}\right] \tag{21}$$

$$A_2 = \frac{m_{20}}{c_2} \exp\left[\frac{|\Delta \mathbf{v}_{20}|}{c_2}\right] \tag{22}$$

It is observed that relation (19) in general does not lead to the conclusion that addition of an impulse at a point where the primer vector magnitude is larger than unity would reduce the cost. In other words, the solution may be optimal even though the primer vector magnitude associated with one of the trajectories exceeds unity for a finite time interval. Conversely, the fact that the primer vector magnitude for one of the vehicles' trajectories never exceeds unity does not necessarily imply that the cooperative solution is optimal and could not be improved by addition of an impulse to that trajectory.

Therefore, depending on which spacecraft's trajectory is considered in a given problem, the time history of the appropriate switching function corresponding to one of the given four cases is calculated to determine if and where along the trajectory addition of an impulse would lower the cost.

Results

The method just outlined was used to generate a family of optimal solutions. The units have been normalized such that the gravity constant $\mu = 1$. The initial orbits of the two spacecraft are both circular with spacecraft 1 having a radius $R_1 = 1$. This will make the corresponding orbital speed $v_1 = 1$ and period $T_1 = 2\pi$. Other problem parameters are R_2, the radius of the initial orbit of spacecraft 2; c_1 and c_2, the magnitudes of the effective exhaust velocities of rocket engines of spacecrafts 1 and 2, respectively; m_{10} and m_{20}, the initial masses of spacecrafts 1 and 2, respectively; β_1 and β_2, the initial fuel mass fractions of spacecraft 1 and 2, respectively; and γ, the angle by which spacecraft 2 leads spacecraft 1 at time $t = 0$.

Optimal Rendezvous Without Fuel Constraints

Table 2 provides a comparison of the cost (in Δm) of cooperative vs noncooperative rendezvous for various values of time of flight (TOF) and angle γ by which vehicle 2 leads vehicle 1 in their circular orbits at $t = 0$. For this comparison we have chosen

$$m_{10} = m_{20} = 10.0, \qquad c_1 = c_2 = 6.0$$
$$\beta_1 = \beta_2 = 0.9, \qquad R_2 = 1.2\ (R_1 = 1)$$

The noncooperative trajectories were found by Chiu[9] and always have vehicle 2 passive whereas vehicle 1 may do zero, one, or two midcourse maneuvers. All of the cases are seen to use less than the 9.0 units of fuel aboard each vehicle, thus fuel is not a constraint. Results for interesting cases with comparatively brief times of flight are shown in Table 2. Cooperative rendezvous is always less costly, sometimes significantly so.

Optimal Rendezvous with Fuel Constraints

Table 3 summarizes the optimal cost (Δm), namely, the total mass of propellant needed for the rendezvous for a family of solutions in which the TOF and the two spacecrafts' initial fuel mass fractions β_1 and β_2 were varied as other problem parameters were fixed. The rendezvous type is defined in Table 1.

The problem parameters are as follows:

$$m_{10} = m_{20} = 10.0, \qquad c_1 = c_2 = 1.0$$
$$R_1 = 1.0, \qquad R_2 = 1.2, \qquad Y = \pi/2$$

The time of flight is expressed as a fraction of the period T_1.

Several observations can be made with regard to Table 3. For any given values of β_1 and β_2, the cost goes down as TOF increases. This is the usual result of relaxing a constraint in a constrained optimization problem. However, unlike the results in the previous section, the optimal rendezvous does not necessarily become noncooperative for large values of TOF. This is the result of the presence of a fuel constraint on at least spacecraft 1 for most values of TOF. The fuel constraint may vanish (become 'inactive') for large enough values of TOF. This is because relaxing the time constraint in the problem, i.e., increasing TOF, may lower the optimal cost to the point where enough fuel exists to complete the rendezvous. This is true for most of the cases considered in Table 3 as for TOF = 1.4 the solution eventually becomes noncooperative, and a Hohmann transfer. (For the present initial geometry, the minimum time required for a Hohmann transfer is about 1.365.) However, if the fuel constraint is severe enough, the rendezvous may remain cooperative for even large values of TOF. This is the case when $\beta_1 = 0.05$ and $\beta_2 = 0.30$. What happens in this case is that the optimal solution still contains a Hohmann transfer on the part of spacecraft 1 with spacecraft 2 remaining in its initial orbit. On interception, due to the lack of sufficient fuel aboard space-

Table 2 Comparison of cost of cooperative and noncooperative rendezvous as a function of time of flight and initial lead angle

γ	TOF	Coop. cost	Noncoop. cost	Saving, %
$\pi/2$	0.3	1.7940	1.8796	4.55
$\pi/2$	0.6	0.5040	0.6048	16.67
$\pi/2$	0.8	0.3019	0.3374	10.52
π	0.3	2.2185	2.3454	5.41
π	0.4	1.4468	1.5715	7.94
π	0.6	0.8445	0.8773	3.74
$3\pi/2$	0.4	1.4511	2.8253	48.64
$3\pi/2$	0.6	0.7936	2.0507	61.30
$3\pi/2$	0.8	0.5694	1.4526	60.80
$3\pi/2$	1.0	0.4691	1.0027	53.22

Table 3 Cost (Δm) vs time of flight and fuel mass fraction

β TOF	0.4	0.6	0.8	1.0	1.2	1.4
β1 = 0.30	5.885[a]	2.770	1.726	1.312	0.950	0.833
β2 = 0.30	3[b]	1	1	5*M*	2	5
β1 = 0.29	5.885	2.770	1.726	1.312	0.950	0.833
β2 = 0.30	[a]	1	1	5*M*	2	5
β1 = 0.25	*[c]	2.770	1.726	1.312	0.950	0.833
β2 = 0.30		1	1	5*M*	2	5
β1 = 0.20	*	2.814	1.726	1.312	0.950	0.833
β2 = 0.30		3	1	5*M*	2	5
β1 = 0.15	*	2.832	1.726	1.312	0.950	0.833
β2 = 0.30		2	1	5*M*	2	5
β1 = 0.05	*	*	1.730	1.312	0.950	0.891
β2 = 0.30			2	5*M*	2	3

[a]First line of each column indicates Δm.
[b]Second line of each column indicates rendezvous type.
[c]Asterisk denotes that a rendezvous solution is infeasible.

Table 4 Cost (Δm) vs time of flight and vehicle initial mass

m_0 TOF	0.4	0.6	0.8	1.0	1.2
m_{10} = 10.0	5.468[a]	2.770	1.726	1.312	0.950
m_{20} = 10.0	5[b]	1	1	5*M*	6*m*
m_{10} = 20.0	7.275	3.741	2.166	1.441	0.950
m_{20} = 10.0	6	2	2	6*m*	6*m*
m_{10} = 30.0	7.275	4.532	2.564	1.441	0.950
m_{20} = 10.0	6	2	2*m*	6*m*	6*m*
m_{10} = 10.0	5.468	3.151	1.861	1.312	1.016
m_{20} = 20.0	5	5	5*M*	5*M*	5*M*
m_{10} = 10.0	5.468	3.151	1.861	1.312	1.016
m_{20} = 30.0	5	5	5*M*	5*M*	5*M*

[a]First line of each column indicates Δm.
[b]Second line of each column indicates rendezvous type.

craft 1, spacecraft 2 is forced to take part in the terminal maneuver. Therefore, unlike a Hohmann transfer, the final orbit of the two spacecraft is different from the initial orbit of spacecraft 2. In this sense, the solution could be considered a pseudo-Hohmann transfer. Because of the presence of a fuel constraint, the optimal cost ($\Delta m = 0.891$) is larger than that of a Hohmann transfer ($\Delta m = 0.833$). Table 3 also indicates that an optimal noncooperative rendezvous is not restricted to large values of TOF, as is the case for TOF = 1.0.

For any given value of TOF, decreasing the amount of fuel on board one of the spacecraft causes the optimal cost to eventually increase. This is consistent with the general rule in optimization that a more rigid constraint will result in a nondecreasing optimal cost. The determining factor in this case is the nature of the optimal terminal maneuver. For instance, in Table 3 consider the values of optimal cost for TOF = 0.6 as the value of β_1 is gradually decreased. For large enough values of β_1, spacecraft 1, which happens to be the more efficient vehicle at the time of interception, has enough propellant to perform the terminal maneuver by itself ($\beta_1 = 0.30$–0.25). However, for a small enough value of β_1, such as $\beta_1 = 0.20$, the terminal maneuver has to include spacecraft 2. As far as the optimal initial burns and the intercepting trajectories are concerned, they are all the same as β_1 is varied, since the initial mass of the vehicle does not vary with β_1. However, the change in the way the terminal maneuver takes place affects both the final orbit of the two vehicles after rendezvous and the optimal cost of the rendezvous. Decreasing the value of β_1 even further may have several consequences.

Spacecraft 1 may still have some fuel left to contribute to the terminal maneuver, and it may do so. But according to the analysis by Prussing and Conway,[7] it may be beneficial for the vehicle that is less efficient at the moment of interception (spacecraft 2 in this case) to perform the terminal maneuver entirely by itself. This may seem unreasonable. The explanation is that the initially less efficient spacecraft 2 will use so much of its fuel in performing the terminal maneuver alone that it becomes more efficient than spacecraft 1 is at the moment of interception and expends sufficient propellant while it is more efficient that the rendezvous is accomplished less expensively than if spacecraft 1 had first exhausted its fuel before spacecraft 2 was used. This is the case for $\beta_1 = 0.15$ and $\beta_2 = 0.30$ ($\Delta m = 2.832$). Another possibility for small enough values of β_1 and β_2 is that the two spacecraft combined may not have enough propellant to perform a rendezvous. In other words, there exists no feasible rendezvous solution. For example, this is the case for $\beta_1 = 0.05$ and $\beta_2 = 0.30$.

On the other hand, considering another value of TOF, the two spacecraft may have enough combined propellant for a rendezvous, however, the fuel constraint for one of the vehicles may be active at the time of its initial burn. This is the case for TOF = 0.8, $\beta_1 = 0.05$, and $\beta_2 = 0.30$. It results in spacecraft 1 expending all its fuel during its initial burn.

Optimal Rendezvous Solution with Different Initial Masses

Table 4 summarizes the optimal cost (Δm) vs TOF for a family of solutions in which the initial masses of the two vehicles, m_{10} and m_{20} were varied. The other problem parameters that were fixed are as follows:

$$R_1 \text{ p}= 1.0, \qquad R_2 = 1.2, \qquad c_1 = c_2 = 1.0$$
$$\beta_1 = \beta_2 = 0.9, \qquad \gamma = \pi/2$$

No fuel constraint has been imposed to avoid interfering with the effects of different values of initial masses. For any given value of TOF, increasing the initial mass of one of the vehicles will eventually result in an optimal noncooperative rendezvous in which the less massive spacecraft is the active vehicle. The critical initial mass ratio of the two vehicles at which this transition occurs depends on the value of TOF, the initial relative geometry of the vehicles, and the efficiency of their rocket engines (the values for c_1 and c_2). For example, in Table 4 for TOF = 0.6 and 0.8 the optimal solution remains cooperative even as spacecraft 1 becomes three times as massive as spacecraft 2. As spacecraft 1 becomes more massive, the geometry of the solution (including the magnitudes of Δv involved) remains the same. It is known from the rocket equation that the mass of the fuel expended for a given velocity change is directly proportional to the initial mass. Therefore, by doubling the initial mass of spacecraft 1, the mass of the fuel required to accomplish the share of maneuvers performed by spacecraft 1 as part of a cooperative maneuver also doubles. This will in turn increase the overall cost of the rendezvous. Yet, because of the particular geometry of the problem, this increase in cost still does not justify the removal of spacecraft 1 from the cooperative rendezvous. In other words, it is still more expensive for the less massive spacecraft 2 to perform the rendezvous by itself. For the same reason, however, doubling the initial mass of spacecraft 2 will force the optimal solution to become noncooperative with spacecraft 1 as the active vehicle.

The results from Table 4 indicate that although the increase of disparity in initial mass of the two spacecraft will eventually make the non-cooperative rendezvous the least expensive solution, the cooperative rendezvous may be the optimal solution for surprisingly large initial mass ratios.

Summary

A method that utilizes the primer vector theory and analytical expressions for gradient of cost has been developed for determining optimal trajectories for the fixed-time cooperative rendezvous of two spacecraft. The cost is the total amount of propellant expended. The analysis is capable of considering arbitrary initial orbits for the two spacecraft and can be used for three-dimensional cases, though only two-dimensional cases are used here as examples. The analysis also has the capability of adding an additional impulse to one of the spacecraft's trajectories, should that result in a lower optimal cost. The point in the trajectory where the midcourse impulse should be added has been determined by a method analogous to that used by Lion and Handelsman,[4] i.e., using the primer vector of the nominal trajectory. However, the necessary conditions derived, which depend on which vehicle is performing the terminal maneuver and to which vehicle's trajectory the midcourse impulse is to be applied, are not the same as the optimality conditions derived by Lion and Handelsman. Fuel constraints imposed on the two spacecraft can be accommodated. Based on the results generated by this work, cooperative rendezvous may yield propellant savings over active/passive rendezvous, especially for cases with relatively brief times of flight. Cooperative rendezvous also expands the range of initial conditions from which rendezvous is feasible, compared to active/passive rendezvous. The work has obvious application to time-critical rendezvous missions such as space rescue.

Appendix

The state transition matrix as determined by Glandorf[8] is provided here for convenience. The 6 × 6 state transition matrix, $\Phi(t_2, t_1)$ may be obtained as

$$\Phi(t_2, t_1) = \Lambda(t_2)\,\Lambda^{-1}(t_1)$$

where t_1 and t_2 are the two times of interest and $\Lambda(t_2)$ and $\Lambda^{-1}(t_1)$ are defined as follows:

$$\Lambda(t_2) = \begin{bmatrix} F_1\mathbf{r} - s\mathbf{v} & F_2\mathbf{r} - q\mathbf{v} & 2\mathbf{r} - F_6\mathbf{v} & \mathbf{v} & F_1\mathbf{h} & F_2\mathbf{h} \\ F_8\mathbf{r} - F_1\mathbf{v} & F_9\mathbf{r} - F_2\mathbf{v} & F_7\mathbf{r} - \mathbf{v} & -F_5\mathbf{r} & F_3\mathbf{h} & F_4\mathbf{h} \end{bmatrix}$$

where

$$q = -(p/h)(p + r)\, r\cos(f)$$

$$s = \frac{(p/h)(p + r)\, r\sin(f) - 3ept}{1 - e^2}$$

$$F_1 = r\cos(f), \qquad F_2 = r\sin(f), \qquad F_3 = -(h/p)\sin(f)$$

$$F_4 = (h/p)[e + \cos(f)], \qquad F_5 = (\mu/r^3), \qquad F_6 = 3t$$

$$F_7 = (3\mu t/r^3), \qquad F_8 = F_3 + sF_5, \qquad F_9 = F_4 + qF_5$$

and $r = |\boldsymbol{r}|$, $\boldsymbol{h} = \boldsymbol{r} \times \boldsymbol{v}$ is the angular momentum vector, e is the eccentricity, and p is the semilatus rectum. Then,

$$\Lambda^{-1}(t_1) = \frac{1}{h^3}\begin{bmatrix} F_2F_5\boldsymbol{\sigma} - F_4\boldsymbol{\omega} & 2F_4\boldsymbol{\sigma} - F_2\boldsymbol{\omega} \\ -F_1F_5\boldsymbol{\sigma} - F_3\boldsymbol{\omega} & -2F_3\boldsymbol{\sigma} + F_1\boldsymbol{\omega} \\ h\boldsymbol{\omega} & -h\boldsymbol{\sigma} \\ -b_1\boldsymbol{\sigma} + b_2\boldsymbol{\omega} & -b_3\boldsymbol{\sigma} + b_4\boldsymbol{\omega} \\ F_4\mathbf{h} & -F_2\mathbf{h} \\ -F_3\mathbf{h} & F_1\mathbf{h} \end{bmatrix}_{t = t_1}$$

where

$$h = |\mathbf{h}|, \qquad \boldsymbol{\sigma} = \mathbf{r} \times \mathbf{h}, \qquad \boldsymbol{\omega} = \mathbf{v} \times \mathbf{h}$$

$$b_1 = h + F_5(F_1q - F_2s), \qquad b_2 = F_6h + F_3q - F_4s$$

$$b_3 = F_6h + 2(F_3q - F_4s), \qquad b_4 = F_1q - F_2s$$

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